European Journal of Mathematics (2022) 8:309–334
https://doi.org/10.1007/s40879-021-00512-6

RESEARCH ARTICLE

MAD families and strategically bounding forcings

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Received: 22 June 2021 / Revised: 1 September 2021 / Accepted: 11 September 2021 /
Published online: 27 January 2022

Abstract

Informally, a proper forcing $\mathbb{P}$ is strategically bounding if there is a strategy to prove that $\mathbb{P}$ is ω^ω-bounding. We prove that certain MAD families are indestructible by strategically bounding forcings. Our motivation for studying this topic is the problem of Roitman: Does $\mathfrak{d} = \omega_1$ imply $\mathfrak{a} = \omega_1$? From this work, it follows that a model of $\omega_1 = \mathfrak{d} < \mathfrak{a}$ cannot be obtained by forcing with a strategically bounding forcing over a model of CH. We prove an iteration theorem for strategically bounding forcings.

Keywords MAD families · Bounding forcings · Strategically bounding · Forcing · Cardinal invariants

Mathematics Subject Classification 03E17 · 03E35 · 03E05

1 Introduction

One of the most intriguing open problems regarding cardinal invariants of the continuum is the following:

The authors gratefully acknowledge support from a CONACyT Grant A1-S-16164 and a PAPIIT Grant IN 104220.

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Problem 1.1 (Roitman) Does $\mathfrak{d} = \omega_1$ imply $\mathfrak{a} = \omega_1$?[1]

A (probably equivalent) version of Roitman's problem is the following:

Problem 1.2 Assume the Continuum Hypothesis (CH) holds in V. Let $\mathcal{A}$ be an MAD family. Is there a proper ω^ω-bounding forcing that destroys $\mathcal{A}$?

Using well-known iteration theorems, it is easy to see that a positive answer to the problem would yield a negative answer to the problem of Roitman. In order to solve it, we must understand which MAD families survive certain forcing extensions. Indestructibility of MAD families and ideals has been thoroughly studied recently. The interested reader may consult [9,12,20–23,29,30] or [7] among many others.

In order to provide a partial answer to Problem 1.2 (and hopefully, to shed some light on Roitman's problem) we restrict our attention to a particular class of ω^ω-bounding forcings—the class of *stratigically bounding forcings* (defined below). One of the main results in this note is that Problem 1.2 has a negative answer for forcings in this class.

Let $\mathbb{P}$ be a partial order and $p \in \mathbb{P}$. The *bounding game* $\mathcal{BG}(\mathbb{P}, p)$ is an infinite two-player game defined as follows:

I	D_0		D_1		$\dots$
II		B_0		B_1	$\dots$

Two players I and II take turns playing subsets of $\mathbb{P}$, player I sets $D_n \subseteq \mathbb{P}$ open dense below p, and player II finite sets $B_n \subseteq D_n$. Player II *wins the game* if there is $q \leqslant p$ such that B_n is predense below q for every $n \in \omega$ (i.e. if every $r \leqslant q$ is compatible with an element of B_n), otherwise player I wins.

Recall that a forcing $\mathbb{P}$ is ω^ω*-bounding* if it does not add unbounded reals. In other words, if $\omega^\omega \cap V$ is still a dominating family after forcing with $\mathbb{P}$. The following is a result of Jech and Zapletal (see [27] and [44, Theorem 3.10.7]):

Proposition 1.3 (Jech, Zapletal) *Let $\mathbb{P}$ be a proper forcing. The following are equivalent:*

(1) *$\mathbb{P}$ is ω^ω-bounding.*
(2) *For every $p \in \mathbb{P}$, the player* I *does not have a winning strategy on $\mathcal{BG}(\mathbb{P}, p)$.*

The main definition of the paper is then natural:

Definition 1.4 Let $\mathbb{P}$ be a partial order. $\mathbb{P}$ is *strategically bounding* if for every $p \in \mathbb{P}$, the player II has a winning strategy on $\mathcal{BG}(\mathbb{P}, p)$.

Examples of strategically bounding forcings are the Sacks, Silver and random forcings. In fact, the usual proofs that these forcings are ω^ω-bounding actually show that they are strategically bounding. Strategically bounding forcings have been studied in the past. In particular, the ccc case has received a lot of attention because of its relation to Maharam's and von Neumann's problems. Let us mention a fundamental result of Fremlin:

[1] The undefined notions will be reviewed in the next section.

Theorem 1.5 (Fremlin, see [1]) *Let $\mathbb{B}$ be a ccc complete Boolean algebra. The following are equivalent:*

(1) $\mathbb{B}$ *is strategically bounding.*
(2) *There is a continuous submeasure on* $\mathbb{B}$.

For more on strategically bounding forcings, the reader may consult [44].

In relation to the problem of Roitman, it is worth pointing out that Shelah proved that the inequality $\mathfrak{d} < \mathfrak{a}$ is consistent ([40], see also [4]). This is achieved by the technique of "iterating along a template". The reader may consult [4–6,14,15,32,33] and [13] to learn more about this topic.

The following problem of Brendle and Raghavan is a weaker version of Roitman's problem:

Problem 1.6 (Brendle, Raghavan [8]) Does $\mathfrak{b} = \mathfrak{s} = \omega_1$ imply $\mathfrak{a} = \omega_1$?

The paper is organized as follows: In Sect. 3 we present the basic theory of strategically bounding forcings and the bounding game. In Sect. 4 we prove that Problem 1.2 has a negative answer when restricted to strategicly bounding forcings. In Sect. 5 we obtain similar results to partitions of compact subsets of ω^ω. In Sect. 6 we prove the preservation of the property of being strategically bounding under countable support iteration. In the last section we present some open questions.

2 Preliminaries

Our notation and definitions are mostly standard, but we will review the main notions used in the paper for the convenience of the reader.

A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* (AD) if the intersection of any two distinct elements of $\mathcal{A}$ is finite, a MAD *family* is an almost disjoint family maximal with respect to inclusion. The *almost disjointness number* $\mathfrak{a}$ is the smallest size of a MAD family.

Given $f, g \in \omega^\omega$, we write $f \leqslant g$ if and only if $f(n) \leqslant g(n)$ for every $n \in \omega$ and $f \leqslant^* g$ if and only if $f(n) \leqslant g(n)$ for all but finitely many $n \in \omega$. A family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leqslant^*$. A family $\mathcal{D} \subseteq \omega^\omega$ is a *dominating family* if for every $f \in \omega^\omega$, there is $g \in \mathcal{D}$ such that $f \leqslant^* g$. The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family and the *dominating number* $\mathfrak{d}$ is the smallest size of a dominating family.

We say that S *splits* X if $S \cap X$ and $X \setminus S$ are both infinite. A family $\mathcal{S} \subseteq [\omega]^\omega$ is a *splitting family* if for every $X \in [\omega]^\omega$ there is $S \in \mathcal{S}$ such that S splits X. The *splitting number* $\mathfrak{s}$ is the smallest size of a splitting family. The reader may consult [3] in order to learn more about the cardinal invariants used in this paper.

Let $\mathcal{I}$ be an ideal on ω, $\mathcal{F}$ a filter on ω and $\mathcal{A}$ a MAD family. Define[2] $\mathcal{I}^+ = \wp(\omega) \setminus \mathcal{I}$, i.e. the subsets of ω that are not in $\mathcal{I}$. We say that a forcing notion $\mathbb{P}$ *destroys* $\mathcal{I}$ if $\mathbb{P}$ adds an infinite subset of ω that is almost disjoint from every element of $\mathcal{I}$. We say that $\mathbb{P}$ *diagonalizes* $\mathcal{F}$ if $\mathbb{P}$ adds an infinite set almost contained in every element of $\mathcal{F}$. It is easy to see that $\mathbb{P}$ destroys $\mathcal{I}$ if and only if $\mathbb{P}$ diagonalizes the filter $\mathcal{I}^* = \{\omega \setminus A \mid A \in \mathcal{I}\}$.

[2] By $\wp(a)$ we denote the powerset of a.

By $\mathcal{I}(\mathcal{A})$ we denote the ideal generated by $\mathcal{A}$ (and the finite sets). We say that $\mathbb{P}$ *destroys* a MAD family $\mathcal{A}$ if $\mathcal{A}$ is no longer maximal after forcing with $\mathbb{P}$, i.e. if and only if $\mathbb{P}$ destroys the ideal $\mathcal{I}(\mathcal{A})$.

Let $T \subseteq \omega^{<\omega}$ be a tree. If $s \in T$ we define $\mathrm{suc}_T(s) = \{\alpha \mid s^\frown\alpha \in T\}$ (where $s^\frown\alpha$ is the sequence that has s as an initial segment and α in the last entry). We say that $f \in \omega^\omega$ is a *branch of* T if $f \restriction n \in T$ for every $n \in \omega$. The set of all branches of T is denoted by $[T]$. For every $n \in \omega$ we define $T_n = \{s \in T \mid |s| = n\}$. If $s \in \omega^{<\omega}$ then the *cone* of s is defined as $\langle s \rangle = \{f \in \omega^\omega \mid s \subseteq f\}$.

All games in the paper are of length ω and we refer to the players simply as player I and player II. We will refer to the player II as a woman and player I as a man.

If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leqslant \delta \rangle$ is a forcing iteration, $\alpha \leqslant \delta$ and $G \subseteq \mathbb{P}_\delta$ is a $(V, \mathbb{P}_\delta)$-generic filter, then G_α denotes $\mathbb{P}_\alpha \cap G$, which is a $(V, \mathbb{P}_\alpha)$-generic filter. Moreover, we will write V_α for $V[G_\alpha]$.

Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leqslant \delta \rangle$ be a countable support iteration. If $\alpha \leqslant \beta \leqslant \delta$ and $G \subseteq \mathbb{P}_\alpha$ is a $(V, \mathbb{P}_\alpha)$-generic filter, in the extension $V[G]$ we define the forcing $\mathbb{P}_\beta/G = \{p \restriction [\alpha, \beta) \mid p \in \mathbb{P}_\beta \wedge p \restriction \alpha \in G\}$. In case we do not need to mention the filter G, we will simply denote this partial order as $\mathbb{P}_\beta/\mathbb{P}_\alpha$. It is known that $\mathbb{P}_\beta$ and $\mathbb{P}_\alpha * (\mathbb{P}_\beta/\mathbb{P}_\alpha)$ are forcing equivalent.

3 The bounding game

In this section, we will study some of the basic properties of the bounding game and strategically bounding forcings (as defined in the introduction). Obviously, every σ-closed forcing is strategically bounding. As mentioned before the Sacks, Silver and random forcings are also strategically bounding. In fact, many definable ω^ω-bounding forcings are strategically bounding by the following result of Zapletal (see [44, Theorem 3.10.7]):

Proposition 3.1 (Zapletal) *Let $\mathbb{P}$ be a proper ω^ω-bounding forcing:*

(1) *If suitable large cardinals exist and $\mathbb{P}$ is universally Baire, then $\mathbb{P}$ is strategically bounding.*
(2) *If $\mathbb{P}$ is of the form* $\mathrm{Borel}(2^\omega)/\mathcal{I}$ *where $\mathcal{I}$ is a σ-ideal on a Polish space that is Π^1_1 on Σ^1_1, then $\mathbb{P}$ is strategically bounding.*[3]

In this way, we can see that there are many examples of strategically bounding forcings. We will first look at some simple variations of the bounding game:

Let $\mathbb{P}$ be a partial order and $p \in \mathbb{P}$. We define $\mathcal{BG}_{\mathrm{anti}}(\mathbb{P}, p)$ ($\mathcal{BG}_{\mathrm{dense}}(\mathbb{P}, p)$, $\mathcal{BG}_{\mathrm{predense}}(\mathbb{P}, p)$) as follows:

I	D_0		D_1		...
II		B_0		B_1	...

[3] The definitions of the undefined notions in this proposition can be consulted in [44] and will not be needed in this note.

where each $D_n \subseteq \mathbb{P}$ is a maximal antichain (dense, predense[4]) below p and $B_n \in [D_n]^{<\omega}$. Player II *wins the game* if there is $q \leqslant p$ such that B_n is predense below q for every $n \in \omega$. As expected, the games are equivalent:

Lemma 3.2 *Let $\mathbb{P}$ be a partial order and $p \in \mathbb{P}$. The following are equivalent:*

(1) *Player* II *has a winning strategy in* $\mathcal{BG}(\mathbb{P}, p)$.
(2) *Player* II *has a winning strategy in* $\mathcal{BG}_{\text{predense}}(\mathbb{P}, p)$.
(3) *Player* II *has a winning strategy in* $\mathcal{BG}_{\text{anti}}(\mathbb{P}, p)$.
(4) *Player* II *has a winning strategy in* $\mathcal{BG}_{\text{dense}}(\mathbb{P}, p)$.

Proof We will first prove that item (1) implies item (2). Given $E \subseteq \mathbb{P}$, denote $E^{\downarrow} = \{r \in \mathbb{P} \mid \exists q \in E\ (r \leqslant q)\}$. We know that if E is predense, then $E^{\downarrow}$ is an open dense set. If player II has a winning strategy for $\mathcal{BG}(\mathbb{P}, p)$, she can obtain a winning strategy in $\mathcal{BG}_{\text{predense}}(\mathbb{P}, p)$ as follows:

If at step n of the game, player I plays E in $\mathcal{BG}_{\text{predense}}(\mathbb{P}, p)$, player II will pretend she is playing the game $\mathcal{BG}(\mathbb{P}, p)$ and player I played $E^{\downarrow}$. If her response (in $\mathcal{BG}(\mathbb{P}, p)$) is $\{a_1, \ldots, a_n\} \subseteq E^{\downarrow}$, for every $i \leqslant n$ she will choose $e_i \in E$ with $a_i \leqslant e_i$ and play $\{e_1, \ldots, e_n\} \subseteq E$ as her response in $\mathcal{BG}_{\text{predense}}(\mathbb{P}, p)$. It is easy to see that this is a winning strategy.

The fact that item (2) implies item (3) is trivial, since every maximal antichain is predense. In order to prove that item (3) implies item (4), it is enough to note that every dense set contains a maximal antichain. Finally, it is clear that item (4) implies item (1). $\square$

From now on, we will only write $\mathcal{BG}$ but use the version of the game most convenient for the problem at hand.

Frequently, one can find a stronger version of strategic bounding, which is the following:

Definition 3.3 Let $\mathbb{P}$ be a partial order. We say that $\mathbb{P}$ is *axiom A for* $\mathfrak{d}$ (or *has an axiom A structure for* $\mathfrak{d}$) if there is a sequence of partial orders $\langle \leqslant_n \rangle_{n \in \omega}$ with the following properties:

- If $p \leqslant_0 q$ then $p \leqslant q$.
- If $p \leqslant_{n+1} q$ then $p \leqslant_n q$ for every $n \in \omega$.
- (*Fusion*) If $\langle p_n \rangle_{n \in \omega}$ is a sequence such that $p_{n+1} \leqslant_n p_n$ for every $n \in \omega$, then there is $q \in \mathbb{P}$ such that $q \leqslant_n p_n$ for every $n \in \omega$.
- (*Bounding Freezing*) For every $p \in \mathbb{P}$, $A \subseteq \mathbb{P}$ a maximal antichain and $n \in \omega$, there is $q \leqslant_n p$ such that $\{r \in A \mid r \text{ and } q \text{ are compatible}\}$ is finite.

Clearly if $\mathbb{P}$ is Axiom A for $\mathfrak{d}$, then $\mathbb{P}$ is strategically bounding. The Axiom A forcings for $\mathfrak{d}$ are also called "Axiom B forcings" (where "B" is for bounding of course). We choose the name axiom A for $\mathfrak{d}$ since it is the natural variation of Axiom A in order to preserve $\mathfrak{d}$ (in a similar way, we could define other notions, like axiom A

[4] Recall that a set A is *predense below a condition* p if for every $q \leqslant p$, there is $r \in A$ that is compatible with q. Equivalently, A is predense if $p \Vdash$ "$A \cap \dot{G} \neq \varnothing$" (where $\dot{G}$ is the canonical name for the generic filter).

for $\mathsf{cof}(\mathcal{N})$, which would be the natural variation of Axiom A for the Sacks property). Very recently, Calderón used Axiom A forcings for $\mathfrak{d}$ in order to solve a long-standing question due to Bartoszyński and Judah (see [10]).

The following result is easy:

Lemma 3.4 *If $\mathbb{P}$ is strategically bounding, then $\mathbb{P}$ is proper and ω^ω-bounding.*

Proof Let $\mathbb{P}$ be a strategically bounding forcing. It is easy to see that $\mathbb{P}$ is ω^ω-bounding, we will prove that it is proper. Let M be a countable elementary submodel with $\mathbb{P} \in M$. Let $p \in M \cap \mathbb{P}$ and choose $\sigma \in M$ a winning strategy for player II in $\mathcal{BG}(\mathbb{P}, p)$. Let $\{D_n \mid n \in \omega\}$ be the collection of all open dense subsets of $\mathbb{P}$ that are in M. Consider the run of $\mathcal{BG}(\mathbb{P}, p)$ in which player I plays D_n at the step n of the game and player II is following σ. Note that every response of player II is a finite subset of $\mathbb{P} \cap M$. Let $q \leqslant p$ be the condition obtained by the victory of player II. It is easy to see that q is an $(M, \mathbb{P})$-generic condition. ◻

We now provide an example of a proper ω^ω-bounding forcing that is not strategically bounding. Although this can be deduced from Theorem 1.5, a direct proof helps to gain more insight into strategically bounding forcings.

Proposition 3.5 *If T is a Suslin tree, then T is a proper ω^ω-bounding forcing that is not strategically bounding.*

Proof It is well known that Suslin trees are ccc and ω-distributive, so in particular they are proper ω^ω-bounding. We argue by contradiction, so assume that T is a Suslin tree that is strategically bounding. Let σ be a winning strategy for player II in the game $\mathcal{BG}(\mathbb{P}, s_0)$, where s_0 is the root of T. In this proof, we will use the version of the bounding game where player I is playing maximal antichains.

Let $a(T)$ denote the partial order of finite antichains of T and order it by inclusion. It is well known that $a(T)$ is a ccc partial order (see e.g. [26] or [43]). Let M be a countable elementary submodel such that $T, \sigma \in M$, let $\delta = M \cap \omega_1 \in \omega_1$, and enumerate $T_\delta = \{t_n \mid n \in \omega\}$[5]. ◻

Claim *Let $n \in \omega$, $\beta < \delta$, $p \in M$ be a partial play of $\mathcal{BG}(\mathbb{P}, s_0)$ in which player* II *is following her strategy and it is the turn of player* I*. There is α with the following properties:*

(1) $\beta < \alpha < \delta$.
(2) *t_n is incompatible with every element of $\sigma(p^\frown T_\alpha)$ (recall that $\sigma(p^\frown T_\alpha)$ is a finite subset of T_α).*

In order to prove the claim, let $Z = \{\sigma(p^\frown T_\alpha) \mid \beta < \alpha\}$. Since Z is an uncountable subset of $a(T)$, we can find $\alpha < \gamma$ such that $\sigma(p^\frown T_\alpha)$ and $\sigma(p^\frown T_\gamma)$ are incompatible, which just means that $\sigma(p^\frown T_\alpha) \cup \sigma(p^\frown T_\gamma)$ is an antichain. Furthermore, by elementarity, we can assume that $\alpha, \gamma \in M$ (i.e. $\alpha, \gamma < \delta$). Since $\sigma(p^\frown T_\alpha) \cup \sigma(p^\frown T_\gamma)$ is an antichain, we know that either t_n is incompatible with every element of $\sigma(p^\frown T_\alpha)$ or with every element of $\sigma(p^\frown T_\gamma)$ (perhaps both). This finishes the proof of the claim.

5 If T is a tree and α is an ordinal, T_α denotes the elements of T of height α.

By the claim, it is possible for player I to play a match of the game

I	T_{α_0}		T_{α_1}		...
II		B_0		B_1	...

so that

1. $\langle\alpha_n\rangle_{n\in\omega}$ is an increasing sequence with limit δ,
2. each B_n was played according to the strategy σ, and
3. t_n is incompatible with every element of B_n (for every $n \in \omega$).

Since σ is a winning strategy for player II, there must be $s \in T$ such that each B_n is a maximal antichain below s. But by item 1 above, it follows that the height of s is at least δ, so s must extend an element of each B_n, but this is a contradiction by item 2 above. □

In particular, it follows that ω^ω-bounding and ccc does not imply strategically bounding. An example of a proper ω^ω-bounding forcing adding reals which is not strategically bounding was used by A. Miller in [42]. We shall discuss this forcing in Sect. 5.

4 Indestructibility of ideals and MAD families

In this section, we will find a family of ideals which cannot be destroyed by a strategically bounding forcing. With this, we will be able to answer Problem 1.2 for the class of strategically bounding forcings. We will need the following game designed by Claude Laflamme ([31]):

Given an ideal $\mathcal{I}$ on ω, define the game $\mathcal{L}(\mathcal{I})$ between players I and II as follows:

I	A_0		A_1		...	
II		s_0		s_1	...	$\bigcup s_n \in \mathcal{I}^+$

At round $n \in \omega$ player I plays $A_n \in \mathcal{I}$ and II responds with $s_n \in [\omega \setminus A_n]^{<\omega}$. Player II wins if $\bigcup s_n \in \mathcal{I}^+$.

Definition 4.1 Let $\mathcal{I}$ be an ideal on ω. We say that $\mathcal{I}$ is *Shelah–Steprāns* if II does not have a winning strategy in $\mathcal{L}(\mathcal{I})$.

In the forthcoming [7] with Brendle and Raghavan we found a simpler characterization of this notion. Given an ideal $\mathcal{I}$ on ω, by $(\mathcal{I}^{<\omega})^+$ we denote the set of all $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ such that for every $A \in \mathcal{I}$ there is $s \in X$ such that $s \cap A = \emptyset$.

Theorem 4.2 (Brendle, Raghavan [7]) *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

(1) *$\mathcal{I}$ is a Shelah–Steprāns ideal.*
(2) *For every $X \in (\mathcal{I}^{<\omega})^+$ there is $Y \in [X]^\omega$ such that $\bigcup Y \in \mathcal{I}$.*[6]

[6] In fact, item (2) of the theorem is the original definition of Shelah–Steprāns ideals. This notion was introduced by Raghavan (under a different name) in [37] .

In other words, an ideal $\mathcal{I}$ is Shelah–Steprāns if and only if for every $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ either there is $A \in \mathcal{I}$ such that $s \cap A \neq \emptyset$ for every $s \in X$ or there is $B \in \mathcal{I}$ containing infinitely many elements of X. Since the paper [7] is still not published, we will avoid making any reference to it in order to make this paper self-contained. Nevertheless, it is worth pointing out that the motivation for this paper comes in part from the work of the authors with Brendle and Raghavan. In [17] the reader can find an application of Shelah–Steprāns ideals.

Definition 4.3 Let $\mathcal{A}$ be a MAD family. We say that $\mathcal{A}$ is *Shelah–Steprāns* if the ideal $\mathcal{I}(\mathcal{A})$ is Shelah–Steprāns.

The notion of Shelah–Steprāns MAD family has its origin in the notion of "strongly separable" introduced by Shelah and Steprāns in [41] where the following is proved (we include a short argument for the sake of completeness):

Proposition 4.4 ([41]) *The Continuum Hypothesis implies that there is a Shelah-Steprāns* MAD *family.*

Proof Let $\mathbb{P}$ be the collection of all countable AD families. If $\mathcal{B}, \mathcal{D} \in \mathbb{P}$, let $\mathcal{B} \leqslant \mathcal{D}$ if $\mathcal{D} \subseteq \mathcal{B}$. If $G \subseteq \mathbb{P}$ is a $(V, \mathbb{P})$-generic filter, define the *generic* MAD *family* $\mathcal{A}_{\text{gen}} = \bigcup G$. It is easy to see that $\mathbb{P}$ is a σ-closed forcing and that $\mathcal{A}_{\text{gen}}$ is forced to be a MAD family. □

Claim *If $G \subseteq \mathbb{P}$ is a generic filter, then the following holds in $V[G]$: For every family $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}(\mathcal{A}_{\text{gen}})^{<\omega})^+$, there is $A \in \mathcal{A}_{\text{gen}}$ such that A contains an element of each X_n.*

To prove the claim, let $\mathcal{B} \in \mathbb{P}$ and $\{\dot{X}_n \mid n \in \omega\}$ be a set of $\mathbb{P}$-names such that $\mathcal{B} \Vdash$ "$\dot{X}_n \in (\mathcal{I}(\dot{\mathcal{A}}_{\text{gen}})^{<\omega})^+$" for every $n \in \omega$. However, since $\mathbb{P}$ is σ-closed we may assume that every $X_n \in V$ and the whole set $\{X_n \mid n \in \omega\}$ is also in V. Let $\mathcal{B} = \{B_n \mid n \in \omega\}$, we recursively define a sequence $\langle s_n \rangle_{n \in \omega}$ of finite sets such that for every $n, m \in \omega$, the following holds:

- $s_n \cap s_m = \emptyset$ if $n \neq m$.
- $s_n \in X_n$.
- $s_n \cap B_i = \emptyset$ for every $i \leqslant n$.

This is easy to see since each X_n is forced to be in $(\mathcal{I}(\dot{\mathcal{A}}_{\text{gen}})^{<\omega})^+$, so in particular they are in $(\mathcal{I}(\mathcal{B})^{<\omega})^+$. Let $A = \bigcup_{n \in \omega} s_n$, it follows that $\mathcal{D} = \mathcal{B} \cup \{A\}$ is an almost disjoint family and it forces the desired conclusion.

Since $\mathbb{P}$ is σ-closed and V is a model of the Continuum Hypothesis, we can find a filter $G \subseteq \mathbb{P}$ such that if $\mathcal{A} = \bigcup G$, the following holds:

- $\mathcal{A}$ is a MAD family.
- For every family $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I}(\mathcal{A})^{<\omega})^+$, there is $A \in \mathcal{A}$ such that A contains an element of each X_n.

Using the second property, we get the following:

(∗) If $T \subseteq ([\omega]^{<\omega})^{<\omega}$ is an $(\mathcal{I}(\mathcal{A})^{<\omega})^+$-branching tree, then there is a branch $\langle s_n \rangle \in [T]$ and $A \in \mathcal{A}$ such that $\bigcup_{n \in \omega} s_n \subseteq A$.

It is easy to see that the property above implies that player II does not have a winning strategy in $\mathcal{L}(\mathcal{A})$. □

An easier proof of the above result can be done with the aid of Theorem 4.2. The reader may note that the proof above gives more, it shows that the generic MAD family added by $\mathbb{P}$ is Shelah–Steprāns. This is only a particular case of a more general result from [7] (where we show that the generic MAD family has a stronger property, which we call "raving"). For more on the existence of Shelah–Steprāns MAD families, the reader may consult [7]. A surprising result of Raghavan is that it is consistent that such families do not exist:

Theorem 4.5 (Raghavan, [37]) *It is consistent with* ZFC *that there are no Shelah–Steprāns* MAD *families.*

We are now ready to prove that Shelah–Steprāns MAD families are indestructible by strategically bounding forcing.

Theorem 4.6 *If $\mathcal{I}$ is a Shelah–Steprāns ideal and $\mathbb{P}$ a strategically bounding forcing, then $\mathbb{P}$ does not destroy $\mathcal{I}$.*

Proof Let $\mathbb{P}$ be strategically bounding and let $\mathcal{I}$ be an ideal on ω such that $\mathbb{P}$ destroys $\mathcal{I}$, we will see that $\mathcal{I}$ is not Shelah–Steprāns. In particular, we will see that player II has a winning strategy in the game $\mathcal{L}(\mathcal{I})$.

Since $\mathbb{P}$ destroys $\mathcal{I}$, there is a $\mathbb{P}$-name $\dot{X}$ for an infinite subset of ω forced to be almost disjoint with every element of $\mathcal{I}$. For every $A \in \mathcal{I}$ and $n \in \omega$, define $D^n_A = \{p \in \mathbb{P} \mid \exists m_p > n\,(p \Vdash \text{“}m_p \in \dot{X} \setminus A\text{”})\}$. It is clear that each D^n_A is an open dense subset of $\mathbb{P}$. We will prove that player II has a winning strategy in $\mathcal{L}(\mathcal{I})$. Fix σ a winning strategy for player II in the bounding game $\mathcal{BG}(\mathbb{P}, 1_{\mathbb{P}})$. While playing the game $\mathcal{L}(\mathcal{I})$, player II will be simulating a game in $\mathcal{BG}(\mathbb{P}, 1_{\mathbb{P}})$ in which she plays as player I.

0) Let $A_0 \in \mathcal{I}$ be the first move of player I in $\mathcal{L}(\mathcal{I})$. Let $B_0 = \sigma(\langle D^0_{A_0}\rangle)$, we know that it is a finite subset of $D^0_{A_0}$. By definition, we know that for every $p \in D^0_{A_0}$, there is $m_p > 0$ such that $p \Vdash \text{“}m_p \in \dot{X} \setminus A_0\text{”}$. Note that, in particular, $m_p \notin A_0$. In this way, player II is allowed to play $s_0 = \{m_p \mid p \in B_0\}$.
1) Let $A_1 \in \mathcal{I}$ be the next move of player I in $\mathcal{L}(\mathcal{I})$. Let $B_1 = \sigma(\langle D^0_{A_0}, D^1_{A_1}\rangle)$, which is a finite subset $D^1_{A_1}$. For every $p \in D^1_{A_1}$, there is $m_p > 1$ such that $p \Vdash \text{“}m_p \in \dot{X} \setminus A_1\text{”}$. We know that $m_p \notin A_1$, so player II is allowed to play $s_1 = \{m_p \mid p \in B_1\}$.

$\vdots$

eral, at step n, player I has played $\langle A_0, \ldots, A_n\rangle$. During the match, player II is secretly building a run $\langle D^0_{A_0}, B_0, \ldots, D^n_{A_n}, B_n\rangle$ in $\mathcal{BG}(\mathbb{P}, 1_{\mathbb{P}})$ in which each B_n was played according to σ. Furthermore, at the l round of the game $\mathcal{L}(\mathcal{I})$, she played $s_l = \{m_p \mid p \in B_m\}$ (where $m_p > l$ and $p \Vdash \text{“}m_p \in \dot{X} \setminus A_p\text{”}$.

$n+1$) Let $A_{n+1} \in \mathcal{I}$ be the next move of player I in $\mathcal{L}(\mathcal{I})$. Let

$$B_{n+1} = \sigma\big(\langle D^0_{A_0}, \ldots, D^{n+1}_{A_{n+1}}\rangle\big),$$

which is a finite subset $D^{n+1}_{A_{n+1}}$. For every $p \in D^{n+1}_{A_{n+1}}$, there is $m_p > n+1$ such that $p \Vdash$ "$m_p \in \dot{X} \setminus A_{n+1}$". Now, player II plays $s_{n+1} = \{m_p \mid p \in B_{n+1}\}$.

I	A_0		A_1		$\ldots$
II		$s_0 = \{m_p \mid p \in B_0\}$		$s_1 = \{m_p \mid p \in B_1\}$	$\ldots$

I	$D^0_{A_0}$		$D^1_{A_1}$		$\ldots$
II		B_0		B_1	$\ldots$

We will prove that player II was the winner in the match of $\mathcal{L}(\mathcal{I})$. In order to prove this, since the side game in $\mathcal{BG}(\mathbb{P}, 1_{\mathbb{P}})$ was played using σ, player II of that simulated match, was the winner. This means that there is $q \in \mathbb{P}$ such that each B_n is predense below q. This implies that $q \Vdash$ "$s_n \cap \dot{X} \neq \varnothing$" for every $n \in \omega$ (recall that $s_n = \{m_p \mid p \in B_m\}$). Let $Y = \bigcup_{n \in \omega} s_n$ it is clear that Y is an infinite set and $q \Vdash$ "$|Y \cap \dot{X}| = \omega$". Since $\dot{X}$ is forced to be almost disjoint from every element of $\mathcal{I}$, it follows that $Y \in \mathcal{I}^+$, which means that player II was the winner of the match. $\square$

In this way, we conclude the following:

Corollary 4.7 (1) *If $\mathcal{A}$ is a Shelah–Steprāns* MAD *family and $\mathbb{P}$ is strategically bounding, then $\mathbb{P}$ does not destroy $\mathcal{A}$.*
(2) *If $V \models$* CH *and $\mathbb{P}$ is a strategically bounding forcing, then $\mathbb{P} \Vdash$ "$\mathfrak{a} = \omega_1$".*

Moreover, later in this paper we will prove that the countable support iteration of strategically bounding forcings is strategically bounding. These two results are useful in computing the almost disjointness number in many forcing extensions.

We will now provide another limitation of strategically bounding forcings. Recall that $\diamondsuit$ is the following statement:

$\diamondsuit$ There is $\mathcal{D} = \{D_\alpha \mid \alpha \in \omega_1\}$ with $D_\alpha \subseteq \alpha$ such that for every $X \subseteq \omega_1$, the set $\{\alpha \mid X \cap \alpha = D_\alpha\}$ is stationary.

In [24] the second author introduced a diamond principle associated to the dominating number:

$\diamondsuit_{\mathfrak{d}}$ There is a sequence $\langle d_\alpha \mid \alpha < \omega_1 \rangle$ where $d_\alpha : \alpha \longrightarrow \omega$ such that for every $f : \omega_1 \longrightarrow \omega$ the set $\{\alpha > \omega \mid f \restriction \alpha \leqslant^* d_\alpha\} \neq \varnothing$. The sequence is called a $\diamondsuit_{\mathfrak{d}}$-*sequence.*

Above, $f \restriction \alpha \leqslant^* d_\alpha$ means that the set $\{\xi < \alpha \mid d_\alpha(\xi) < f(\xi)\}$ is finite. It is easy to see that $\diamondsuit_{\mathfrak{d}}$ implies that $\mathfrak{d} = \omega_1$. The motivation for introducing the principle $\diamondsuit_{\mathfrak{d}}$ was also the problem of Roitman. While it is unknown if $\mathfrak{d} = \omega_1$ suffices to construct a MAD family of size ω_1, it is possible to do it with $\diamondsuit_{\mathfrak{d}}$:

Proposition 4.8 ([24]) $\diamondsuit_{\mathfrak{d}}$ *implies* $\mathfrak{a} = \omega_1$.

In [24] it is proved that forcing with a large measure algebra over a model of $\diamondsuit$ gives a model of $\diamondsuit_{\mathfrak{d}}$. On the other hand, in [16] the first author proved that $\diamondsuit_{\mathfrak{d}}$ holds in the model of [38] (see [16,38] and [11] to learn more about this interesting model). The next theorem generalizes both results.

By LIM(ω_1) we denote the set of all countable limit ordinals. We start with a lemma:

Lemma 4.9 *Let $V \models \diamondsuit$, $\mathbb{P}$ be a forcing notion and κ a large enough regular cardinal. There is a sequence $\langle (M_\alpha, p_\alpha, \dot{f}_\alpha)\rangle_{\alpha \in \mathrm{LIM}(\omega_1)}$ such that for every $\alpha \in \mathrm{LIM}(\omega_1)$ the following holds:*

(1) *M_α is a countable elementary submodel of $\mathsf{H}(\kappa)$ such that $\mathbb{P}, p_\alpha, \dot{f}_\alpha \in M_\alpha$.*
(2) *$p_\alpha \in \mathbb{P}$ and $p_\alpha \Vdash$ "$\dot{f}_\alpha : \omega_1 \longrightarrow \omega$".*

The sequence $\langle (M_\alpha, p_\alpha, \dot{f}_\alpha)\rangle_{\alpha \in \mathrm{LIM}(\omega_1)}$ has the property that for every $p \in \mathbb{P}$ and $\dot{f}$ such that $p \Vdash$ "$\dot{f} : \omega_1 \longrightarrow \omega$", there is a countable $N \preceq \mathrm{H}(\kappa)$ and $\alpha < \omega_1$ such that the following conditions hold:

(1) $\mathbb{P}, p, \dot{f} \in N$.
(2) $M_\alpha \cap \omega_1 = \alpha$.
(3) *The structures $(N, \in, \mathbb{P}, \Vdash_{\mathbb{P}}, p, \dot{f})$ and $(M_\alpha, \in, \mathbb{P}, \Vdash_{\mathbb{P}}, p_\alpha, \dot{f}_\alpha)$ are isomorphic.*[7]

Proof Using $\diamondsuit$ we can find a sequence $\langle \mathfrak{A}_\alpha = (\alpha, \triangleright_\alpha, P_\alpha, \rightsquigarrow_\alpha, r_\alpha, h_\alpha)\rangle_{\alpha \in \mathrm{LIM}(\omega_1)}$ such that for every structure $\mathfrak{A} = (\omega_1, \triangleright, P, \rightsquigarrow, r, h)$ there are stationary many α such that $\mathfrak{A}_\alpha$ is a substructure of $\mathfrak{A}$. Given α a limit ordinal, in case there are a countable $M \preceq \mathsf{H}(\kappa)$, $p \in \mathbb{P}$, $\dot{f}$ such that $\mathbb{P}, p, \dot{f} \in M$, $M \cap \alpha = \alpha$, $p \Vdash$ "$\dot{f} : \omega_1 \longrightarrow \omega$" and $(M, \in, \mathbb{P}, \Vdash_{\mathbb{P}}, p, \dot{f})$ is isomorphic to $\mathfrak{A}_\alpha$ then we choose one of them and define $M_\alpha = M$, $p_\alpha = p$ and $\dot{f}_\alpha = \dot{f}$. If there is no M satisfying those properties, we just take any $(M_\alpha, p_\alpha, \dot{f}_\alpha)$ satisfying the properties (1) and (2). We will now prove $\mathcal{D} = \{(M_\alpha, p_\alpha, \dot{f}_\alpha) \mid \alpha \in \mathrm{LIM}(\omega_1)\}$ has the desired properties.

Let $p \in \mathbb{P}$ and $\dot{f}$ be such that $p \Vdash$ "$\dot{f} : \omega_1 \longrightarrow \omega$". Recursively, we build $\{N_\alpha \mid \alpha < \omega_1\}$ a continuous $\in$-chain of countable elementary submodels of $\mathsf{H}(\kappa)$ such that $p, \dot{f}, \mathbb{P} \in N_0$. Let $N = \bigcup_{\alpha \in \omega_1} N_\alpha$, since N has size ω_1, then we can define a structure $\mathfrak{A} = (\omega_1, \triangleright, P, \rightsquigarrow, r, h)$ that is isomorphic to $(N, \in, \mathbb{P}, \Vdash_{\mathbb{P}}, p, \dot{f})$. Let $F : \omega_1 \longrightarrow N$ be an isomorphism.

It is easy to see that $\{\alpha \in \mathrm{LIM}(\omega_1) \mid N_\alpha \cap \omega_1 = \alpha \wedge F[\alpha] = N_\alpha\}$ is a club. In this way, we can find a limit α such that $F[\alpha] = N_\alpha$, $N_\alpha \cap \omega_1 = \alpha$ and $\mathfrak{A}_\alpha$ is a substructure of $\mathfrak{A}$. Note that N_α, p and $\dot{f}$ satisfy the conditions of the definition at step α, so $(M_\alpha, \in, \mathbb{P}, \Vdash_{\mathbb{P}}, p_\alpha, \dot{f}_\alpha)$ is isomorphic to $\mathfrak{A}_\alpha$ hence it is also isomorphic to $(N, \in, \mathbb{P}, \Vdash_{\mathbb{P}}, p, \dot{f})$. □

We can now prove the theorem:

Theorem 4.10 *Let $V \models \diamondsuit$. If $\mathbb{P}$ is a strategically bounding forcing, then $\mathbb{P} \Vdash$ "$\diamondsuit_{\mathfrak{d}}$".*

Proof Fix a sequence $\langle (M_\alpha, p_\alpha, \dot{f}_\alpha)\rangle_{\alpha \in \mathrm{LIM}(\omega_1)}$ as in Lemma 4.9. We want to define the sequence $\mathcal{D} = \{d_\alpha : \alpha \longrightarrow \omega \mid \alpha < \omega_1\}$. Let $\alpha \in \omega_1$. In case $M_\alpha \cap \omega_1 \neq \alpha$,

[7] In here, a function $g : N \longrightarrow M_\alpha$ is an isomorphism if for every $a, b \in N$, the following conditions hold:

(a) g is bijective.
(b) $a \in b$ if and only if $g(a) \in g(b)$.
(c) $g(\mathbb{P}) = \mathbb{P}$.
(d) $g(p) = p_\alpha$.
(e) $g(\dot{f}) = g(\dot{f}_\alpha)$.
(f) For every $x_1, \ldots, x_n \in N$ and φ a set-theoretic formula, $a \Vdash_{\mathbb{P}}$"$\varphi(x_1, \ldots, x_n)$" if and only if $g(a) \Vdash_{\mathbb{P}}$"$\varphi(g(x_1), \ldots, g(x_n))$".

let d_α be any constant function. Now, fix α such that $M_\alpha \cap \omega_1 = \alpha$, and choose an enumeration $\alpha = \{\alpha_n \mid n \in \omega\}$. Let $\sigma_\alpha \in M_\alpha$ be a winning strategy for $\mathcal{BG}(\mathbb{P}, p_\alpha)$. For every $n \in \omega$, define $D_n^\alpha = \{q \leqslant p_\alpha \mid \exists m\, (q \Vdash \text{“}\dot{f}_\alpha(\alpha_n) = m\text{”})\}$. It is clear that D_n^α is an open dense set below p_α and $D_n^\alpha \in M$. Consider the following run of the game $\mathcal{BG}(\mathbb{P}, p_\alpha)$: □

I	D_0^α		D_1^α		...
II		B_0^α		B_1^α	...

where each B_n^α is played according to the strategy σ_α. Note that although the whole sequence $\langle D_n^\alpha \rangle_{n\in\omega}$ is not in M_α, every initial segment of it is.

For every $q \in D_n^\alpha$, let $m_{\alpha_n}^q \in \omega$ be such that $q \Vdash \text{“}\dot{f}_\alpha(\alpha_n) = m_{\alpha_n}^q\text{”}$. Finally, define the function $d_\alpha : \alpha \longrightarrow \omega$ such that $d_\alpha(\alpha_n) = \max\{m_{\alpha_n}^q \mid q \in B_n^\alpha\} + 1$. Let $\mathcal{D} = \{d_\alpha \mid \alpha \in \omega_1\}$. We will prove that $\mathcal{D}$ is a $\diamondsuit_{\mathfrak{d}}$-sequence after forcing with $\mathbb{P}$.

Let $p \in \mathbb{P}$ and $\dot{f}$ a $\mathbb{P}$-name such that $p \Vdash \text{“}\dot{f} : \omega_1 \longrightarrow \omega\text{”}$. By Lemma 4.9, we know that there is a countable $N \preceq \mathsf{H}(\kappa)$ and $\alpha < \omega_1$ such that the following conditions hold:

1. $\mathbb{P}, p, \dot{f} \in N$.
2. $M_\alpha \cap \omega_1 = \alpha$.
3. The structures $(N, \in, \mathbb{P}, \Vdash_{\mathbb{P}}, p, \dot{f})$ and $(M_\alpha, \in, \mathbb{P}, \Vdash_{\mathbb{P}}, p_\alpha, \dot{f}_\alpha)$ are isomorphic.

Let $H : M_\alpha \longrightarrow N$ be (the unique) isomorphism given by point 3. Recall that $H(\mathbb{P}) = \mathbb{P}$, $H(p_\alpha) = p$ and $H(\dot{f}_\alpha) = \dot{f}$. Recall that if $q \in B_n^\alpha$, then q has the following properties:

1. $q \leqslant p_\alpha$.
2. $q \Vdash \text{“}\dot{f}_\alpha(\alpha_n) = m_{\alpha_n}^q\text{”}$.

Since H is an isomorphism (and countable ordinals are fixed by isomorphisms) it follows that:

1. $H(q) \leqslant p$.
2. $H(q) \Vdash \text{“}\dot{f}(\alpha_n) = m_{\alpha_n}^q\text{”}$.

Furthermore, $H(\sigma_\alpha)$ is a winning strategy in $\mathcal{BG}(\mathbb{P}, p)$[8] and for every $n \in \omega$, the sequence $L_n = \langle H(D_0^\alpha), H(B_0^\alpha), \ldots, H(D_n^\alpha), H(B_n^\alpha)\rangle$ is a legal partial play in $\mathcal{BG}(\mathbb{P}, p)$ in which player II is following $H(\sigma_\alpha)$. Let $L = \bigcup_{n\in\omega} L_n$, since every initial segment of L is a partial play in which player II is following a winning strategy, it follows that L is a run of the game and player II was the winner (note that $L \notin N$, but it does not matter, the important point is that every $L_n \in N$). Since player II was the winner, there is $r \leqslant p$ such that each $H(B_n^\alpha) = H[B_n^\alpha]$ is predense below r. It follows that $r \Vdash \text{“}\dot{f} \restriction \alpha \leqslant \dot{f}_\alpha\text{”}$ and we are done. □

5 Strategically bounding forcings and tree-MAD families

Let $\mathcal{A}$ be a subfamily of $[\omega^{<\omega}]^\omega$. We will say that $\mathcal{A}$ is a *tree*-AD *family* if it is an AD family and every element of $\mathcal{A}$ is a finitely branching tree. We will say that $\mathcal{A}$ is a

[8] Recall that σ_α is a winning strategy in $\mathcal{BG}(\mathbb{P}, p_\alpha)$. Since H is an isomorphism, $H(\mathbb{P}) = \mathbb{P}$ and $H(p_\alpha) = p$, it follows that $H(\sigma_\alpha)$ is a winning strategy in $\mathcal{BG}(\mathbb{P}, p)$.

tree-MAD *family* if it is a maximal tree-AD family. Note that tree-MAD families are not MAD families of $[\omega^{<\omega}]^\omega$ (for example, if $\mathcal{A}$ is a tree-MAD family, then ω^1 is almost disjoint from every element of $\mathcal{A}$).

Definition 5.1 $\mathfrak{a}_T$ is the smallest size of a tree-MAD family.

This cardinal invariant has been studied (although not necessarily by that name) in [19,34,36,42] and [18].

We shall fix some notation first: If $x \in \omega^{\leqslant\omega}$, we denote $\widehat{x} = \{x \restriction n \mid n \in \omega\} \subseteq \omega^{<\omega}$. Furthermore, if $B \subseteq \omega^{\leqslant\omega}$, we define $\widehat{B} = \{\widehat{x} \mid x \in B\}$. The following result is well-known, we prove it here for completeness:

Lemma 5.2 *Let $\mathcal{A}$ be a tree-*AD *family.*

(1) *$\mathcal{A}$ is a tree-*MAD *family if and only if $\{[T] \mid T \in \mathcal{A}\}$ is a partition of ω^ω into compact sets.*
(2) *$\mathfrak{a}_T$ is the smallest size of a partition of ω^ω into compact sets.*

Proof First, assume that $\mathcal{A}$ is a tree-MAD family. Since $\mathcal{A}$ is an AD family, it follows that $[T] \cap [S] = \varnothing$ whenever $T, S \in \mathcal{A}$ and $T \neq S$. Now, if $x \in \omega^\omega$ we have that $\widehat{x}$ is an infinite tree. By the maximality of $\mathcal{A}$, there must be $T \in \mathcal{A}$ such that $\widehat{x} \cap T$ is infinite. Since T is a tree, it follows that $x \in [T]$. In order to prove the second item of the lemma, let $\mathcal{C}$ be a partition of ω^ω in compact sets. It is well known that every compact subset of ω^ω is of the form $[T]$ for some finitely branching T (see [28]). Furthermore, it follows by König's lemma that if T and S are two finitely branching trees, then $T \cap S$ is finite if and only if $[T] \cap [S] = \varnothing$. The result follows by this observation and the first point of the lemma. □

It is well known that $\mathfrak{d}$ is the smallest size of a cover of ω^ω of compact sets (see [2]). From this remark, it follows that $\mathfrak{d} \leqslant \mathfrak{a}_T$. Using a forcing of Miller [34], Spinas proved the following (see [42]):

Theorem 5.3 (Spinas) *There is a model of* ZFC *where $\mathfrak{d} < \mathfrak{a}_T$.*

In contrast with this result, in [35] Džamonja, Moore and the second author proved the following:

Theorem 5.4 *$\diamondsuit_{\mathfrak{d}}$ implies $\mathfrak{a}_T = \omega_1$.*

Let $\mathcal{A}$ be a tree-MAD family and $\mathbb{P}$ a forcing notion. We say that $\mathbb{P}$ *destroys* $\mathcal{A}$ if $\mathcal{A}$ is no longer a tree-MAD family after forcing with $\mathbb{P}$. If $\mathbb{P}$ does not destroy $\mathcal{A}$, we say that $\mathbb{P}$ *preserves* $\mathcal{A}$. By Lemma 5.2, we have the following:

Corollary 5.5 *Let $\mathcal{A}$ be a tree-*MAD *family and $\mathbb{P}$ a partial order. The following are equivalent:*

(1) *$\mathbb{P}$ destroys $\mathcal{A}$.*
(2) *There is $\dot{r}$ a $\mathbb{P}$-name for a branch such that $\mathbb{P} \Vdash$ "$\dot{r} \notin [T]$" for every $T \in \mathcal{A}$.*

In this section we present an analogue of Corollary 4.7 for tree-MAD families.

We need some further notation. If $T \subseteq \omega^{<\omega}$ is a finite tree, by $[T]$ we denote the maximal nodes of T. If $T, S \subseteq \omega^{<\omega}$ are trees, we say that S is an *end-extention of* T (denoted by $T \sqsubseteq S$) if $T \subseteq S$ and every $s \in S \setminus T$ extends an element of $[T]$. Note that if $\{T_n \mid n \in \omega\}$ is a set of finitely branching trees such that $T_n \sqsubseteq T_{n+1}$ for every $n \in \omega$, then $\bigcup T_n$ is a finitely branching tree. We will now introduce the following game:

Let $\mathcal{A}$ be a tree-MAD family. Define the game $\mathcal{L}_T(\mathcal{A})$ between players I and II as follows:

I	A_0		A_1		...
II		L_0		L_1	...

The game is played so that for every $n \in \omega$, the following holds:

1. A_n is the union of finitely many trees of $\mathcal{A}$ (so $A_n \in \mathcal{I}(\mathcal{A})$).
2. L_n is a finitely branching tree such that $[L_n] \cap A_n = \emptyset$.
3. $L_n \sqsubseteq L_{n+1}$.

We will say that player II *won the match* if $\bigcup_{n\in\omega} L_n \in \mathcal{I}(\mathcal{A})^+$.

Definition 5.6 Let $\mathcal{A}$ be a tree-MAD family. We say that $\mathcal{A}$ is *tree Shelah–Steprāns* if player II does not have a winning strategy in the game $\mathcal{L}_T(\mathcal{A})$.

The desired analogue of Corollary 4.7 is the following:

Theorem 5.7 *Let $\mathcal{A}$ be a tree-*MAD *family and $\mathbb{P}$ a strategically bounding forcing. If $\mathcal{A}$ is a tree Shelah–Steprāns family, then $\mathbb{P}$ preserves $\mathcal{A}$.*

Proof Let $\mathbb{P}$ be a strategically bounding forcing and $\mathcal{A}$ a tree-MAD family that is destroyed by $\mathbb{P}$, we will show that $\mathcal{A}$ is not a tree Shelah–Steprāns family (i.e. player II has a winning strategy in the game $\mathcal{L}_T(\mathcal{A})$).

Since $\mathbb{P}$ destroys $\mathcal{A}$, we know that there is a $\mathbb{P}$-name $\dot{r}$ for an element of ω^ω such that $\mathbb{P} \Vdash$ "$\dot{r} \notin [T]$" for every $T \in \mathcal{A}$. Given $p \in \mathbb{P}$ define $z_p = \bigcup\{t \in \omega^{<\omega} \mid p \Vdash$ "$t \subseteq \dot{r}$"$\}$. Since $\dot{r}$ must be the name for a new real, it follows that $z_p \in \omega^{<\omega}$ for every $p \in \mathbb{P}$. For every $A \subseteq \omega^{<\omega}$ that is the union of finitely many trees of $\mathcal{A}$ and $n \in \omega$, define $D^n(A) = \{p \in \mathbb{P} \mid z_p \notin A \wedge |z_p| > n\}$. It is easy to see that each $D^n(A)$ is an open dense subset of $\mathbb{P}$. Let $X \subseteq \mathbb{P}$, define $Z(X) = \{z_p \mid p \in X\}$ and $\widehat{Z}(X) = \widehat{Z(X)}$. Finally, if $Y \in [\mathbb{P}]^{<\omega}$ define $E(Y)$ as the set of all $p \in \mathbb{P}$ such that one of the following conditions holds:

1. p extends an element of Y, or
2. p is incompatible with every element of Y.

It is easy to see that $E(Y)$ is an open dense subset of $\mathbb{P}$. We will now describe a winning strategy for player II in the game $\mathcal{L}_T(\mathcal{A})$. Fix σ a winning strategy for player II in the $\mathcal{BG}(\mathbb{P}, 1_{\mathbb{P}})$. While playing the game $\mathcal{L}_T(\mathcal{A})$, player II will be simulating a game in $\mathcal{BG}(\mathbb{P}, 1_{\mathbb{P}})$ in which she pretends she is the player I for that game.

(0) Let A_0 be the first move of player I in $\mathcal{L}_T(\mathcal{A})$. Let $D_0 = D^0(A_0)$ and $B_0 = \sigma(\langle D_0 \rangle)$. Now, player II plays $L_0 = \widehat{Z}(B_0)$ in $\mathcal{L}_T(\mathcal{A})$.

(1) Let A_1 be the next move of player I in $\mathcal{L}_T(\mathcal{A})$. Let $D_1 = D^1(A_1) \cap E(B_0)$ and $B_1 = \sigma(\langle D_0, D_1 \rangle)$. Let C_1 be the elements of B_1 that extend an element of B_0. Now, player II plays $\widehat{Z}(C_1)$.

$(n+1)$ In general at step n, player I has played $\langle A_0, \ldots, A_n \rangle$. During the match, player II is secretly building a run $\langle D_0, B_0, \ldots, D_n, B_n \rangle$ in $\mathcal{BG}(\mathbb{P}, 1_{\mathbb{P}})$ and $\langle C_0, \ldots, C_n \rangle$ with the following properties:

(a) Each D_n is open dense and the B_n are played following σ.
(b) $D_0 = D^0(A_0)$ and $C_0 = B_0$.
(c) C_{i+1} is the set of elements of B_{i+1} that extend an element of C_i.
(d) $D_{i+1} = D^i(A_i) \cap E(C_i)$.

Furthermore, at the round l of the game $\mathcal{L}_T(\mathcal{A})$, she played $\widehat{Z}(C_l)$.
Now, let A_{n+1} be the next move of player I in $\mathcal{L}_T(\mathcal{A})$. We define the items as above and continue.

We will prove that player II was the winner in the match of $\mathcal{L}_T(\mathcal{A})$. In order to prove this, since the side game in $\mathcal{BG}(\mathbb{P}, 1_{\mathbb{P}})$ was played using σ, player II of that simulated match, was the winner. This means that there is $q \in \mathbb{P}$ such that each B_n is predense below q. Furthermore, by a simple induction this implies that each C_n is predense below q. This implies that $q \Vdash$ "$\widehat{r} \cap [\widehat{Z}(C_n)] \neq \emptyset$" for every $n \in \omega$. Let $L = \bigcup_{n \in \omega} \widehat{Z}(C_n)$, it is clear that Y is a finitely branching tree and $q \Vdash$ "$\dot{r} \in [L]$". Since $\dot{r}$ is forced to not be in the branches of every element of $\mathcal{A}$, it follows that $L \in \mathcal{I}(\mathcal{A})^+$, which means that player II was the winner of the match. □

A similar argument as the one of Proposition 4.4 gives the following:

Proposition 5.8 *The Continuum Hypothesis implies that there is a tree Shelah–Steprāns* MAD *family.*

In this way, we conclude the following:

Corollary 5.9 *If* $V \models$ CH *and* $\mathbb{P}$ *is strategically bounding then* $\mathbb{P} \Vdash$ "$\mathfrak{a}_T = \omega_1$".

With these results, we can get a more interesting example of a proper ω^ω-bounding forcing that is not strategically bounding. The following forcing notion was introduced by Miller in [34]:

Definition 5.10 Let $\mathcal{A}$ be a tree-MAD family. $\mathbb{P}(\mathcal{A})$ is the collection of all p such that the following holds:

- $p \subseteq \omega^{<\omega}$ is a tree such that every $s \in p$ has at most two immediate successors and every node in p can be extended to a splitting node.

- If $T \in \mathcal{A}$, then $[T] \cap [p]$ is nowhere dense in $[p]$.

If $p, q \in \mathbb{P}(\mathcal{A})$, then $p \leqslant q$ if and only if $p \subseteq q$.

In [34] Miller proved that the forcing $\mathbb{P}(\mathcal{A})$ is a proper forcing and that it destroys $\mathcal{A}$. Furthermore, in [42] Spinas proved that $\mathbb{P}(\mathcal{A})$ is ω^ω-bounding. Hence:

Corollary 5.11 *If $\mathcal{A}$ is a tree Shelah–Steprāns* MAD *family, then $\mathbb{P}(\mathcal{A})$ is a proper ω^ω-bounding forcing which is not strategically bounding.*

6 Iteration of strategically bounding forcings

In this section, we will prove that the countable support iteration of strategically bounding forcings is strategically bounding. This is particularly useful when combined with Theorems 4.6 and 4.10. Our proof for the limit case is based on the proof of preservation of properness on the first pages of chapter XII of Shelah's [39]. This proof can probably be used to obtain alternative proofs of other iteration theorems. No previous knowledge of this proof is needed.

We start with a simple observation:

Lemma 6.1 *Let $\mathbb{P}, \mathbb{Q}$ be partial orders such that $\mathbb{P}$ is a dense suborder of $\mathbb{Q}$. If $\mathbb{P}$ is strategically bounding, then $\mathbb{Q}$ is strategically bounding.*

Proof Let $q \in \mathbb{Q}$, we must prove that player II has a winning strategy in $\mathcal{BG}(\mathbb{Q}, q)$. Since $\mathbb{P}$ is dense in $\mathbb{Q}$, we can find $p \in \mathbb{P}$ extending q. Let σ be a winning strategy for player II in the $\mathcal{BG}(\mathbb{P}, p)$. Given $D \subseteq \mathbb{Q}$ an open dense set below q, define $\overline{D} = D \cap \mathbb{P}$. It is easy to see that $\overline{D}$ is open dense below p.

We will now teach player II how to win in the $\mathcal{BG}(\mathbb{Q}, q)$.

(0) Let $D_0 \subseteq \mathbb{Q}$ be the first move of player I. We know that $\overline{D_0}$ is a valid move for player I in $\mathcal{BG}(\mathbb{P}, p)$. Player II will play $B_0 = \sigma(\langle \overline{D_0} \rangle)$ in $\mathcal{BG}(\mathbb{Q}, q)$.
(1) Let $D_1 \subseteq \mathbb{Q}$ be the next move of player I. We know that $\overline{D_1}$ is a valid move for player I in $\mathcal{BG}(\mathbb{P}, p)$. Player II will play $B_1 = \sigma(\langle \overline{D_0}, \overline{D_1} \rangle)$ in $\mathcal{BG}(\mathbb{Q}, q)$.

$\vdots$

$\mathcal{BG}(\mathbb{Q}, q)$

I	D_0		D_1			
II		B_0		B_1		

$\mathcal{BG}(\mathbb{P}, p)$

I	$\overline{D_0}$		$\overline{D_1}$			
II		B_0		B_1		

It is easy to see that this describes a winning strategy for player II in $\mathcal{BG}(\mathbb{Q}, q)$. $\square$

In particular, it follows that if $\mathbb{P}$ is strategically bounding, then its Boolean completion is also strategically bounding. We will also need the following:

Lemma 6.2 *Let $\mathbb{P}$ be a partial order, $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a partial order and $D \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ an open dense set.*

(1) *If $G \subseteq \mathbb{P}$ is a $(V, \mathbb{P})$-generic filter, then (in $V[G]$).*
$D^G = \{\dot{q}[G] \mid \exists\, p \in G\, ((p, \dot{q}) \in D)\}$ *is an open dense subset of* $\dot{\mathbb{Q}}[G]$.
(2) *Let $\dot{X}$ be a $\mathbb{P}$-name for a finite subset of $D^{\dot{G}}$. Define $D^{\mathbb{P},\dot{X}}$ as the set of all $p \in \mathbb{P}$ such there is a set $\{\dot{q}_1, \ldots, \dot{q}_n\}$ for which the following conditions hold:*

(a) $p \Vdash$ "$\dot{X} = \{\dot{q}_1, \ldots, \dot{q}_n\}$".
(b) $(p, \dot{q}_i) \in D$ *for every* $i \leqslant n$.

Then $D^{\mathbb{P},\dot{X}}$ is an open dense subset of $\mathbb{P}$.

Proof The first part of the lemma is well known and may be consulted in [26], the second part is straight forward. □

We can now prove the preservation result for the successor case:

Proposition 6.3 *If $\mathbb{P}$ is strategically bounding and $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is strategically bounding", then $\mathbb{P} * \dot{\mathbb{Q}}$ is strategically bounding.*

Proof Let $(x, \dot{y}) \in \mathbb{P} * \dot{\mathbb{Q}}$, we need to prove that player II has a winning strategy on $\mathcal{BG}(\mathbb{P} * \dot{\mathbb{Q}}, (x, \dot{y}))$. Let σ be a winning strategy for player II in $\mathcal{BG}(\mathbb{P}, x)$ and let $\dot{\pi}$ be the name for a winning strategy for player II in $\mathcal{BG}(\dot{\mathbb{Q}}, \dot{y})$. We define a strategy for player II in the $\mathcal{BG}(\mathbb{P} * \dot{\mathbb{Q}}, (x, \dot{y}))$ as follows:

At step 0, assume that player I plays an open dense set $D_0 \subseteq \mathbb{P} * \dot{\mathbb{Q}}$. By Lemma 6.2, we know that $D_0^{\dot{G}} = \{\dot{q}[\dot{G}] \mid \exists\, p \in \dot{G}\, ((p, \dot{q}) \in D_0)\}$ is forced to be an open dense set of $\dot{\mathbb{Q}}[\dot{G}]$ (where $\dot{G}$ is the $\mathbb{P}$-name for the generic filter). In this way, $\dot{X}_0 = \dot{\pi}(D_0^{\dot{G}})$ is a $\mathbb{P}$-name for a finite subset of $D_0^{\dot{G}}$. Again by Lemma 6.2, we know that $D^{\mathbb{P},\dot{X}_0}$ is an open dense subset of $\mathbb{P}$. Consider $\sigma(D^{\mathbb{P},\dot{X}_0})$. We know that $\sigma(D^{\mathbb{P},\dot{X}_0})$ is a finite set. For every $p \in \sigma(D^{\mathbb{P},\dot{X}_0})$, there is Y_p^0 a finite set of $\mathbb{P}$-names for elements of $\dot{\mathbb{Q}}$ such that $p \Vdash$ "$\dot{X}_0 = \{\dot{q} \mid \dot{q} \in Y_p^0\}$". Finally, player II will play (in $\mathcal{BG}(\mathbb{P} * \dot{\mathbb{Q}}, (x, \dot{y}))$) the set $E_0 = \{(p, \dot{q}) \mid p \in \sigma(D^{\mathbb{P},\dot{X}_0}) \wedge \dot{q} \in Y_p^0\}$. It is easy to see that E_0 is a finite subset of D.

$\mathcal{BG}(\mathbb{P} * \dot{\mathbb{Q}}, (x, \dot{y}))$

I	D_0			
II				

$\mathcal{BG}(\dot{\mathbb{Q}}[G], \dot{y}[G])$

I	$D_0^{\dot{G}}$		
II		X_0	

where $X_0 = \dot{\pi}(D_0^{\dot{G}})$.

$\mathcal{BG}(\mathbb{P}, x)$

I	$D^{\mathbb{P},\dot{X}_0}$		
II		$\sigma(D^{\mathbb{P},\dot{X}_0})$	

In general, at step $n + 1$, player I has played $D_0, \ldots, D_n$ and player II played the sets $E_0, \ldots, E_n$. Also secretly, she has constructed the sets $\langle \dot{X}_0, \ldots, \dot{X}_n \rangle$ such that for every $i \leqslant n$, the following holds:

1. $\dot{X}_i = \dot{\pi}(D_0^{\dot{G}}, \ldots, D_i^{\dot{G}})$.
2. For every $p \in \sigma(D_0^{\mathbb{P},\dot{X}_0}, D_1^{\mathbb{P},\dot{X}_1}, \ldots, D_i^{\mathbb{P},\dot{X}_i})$, there is a finite set Y_p^i of $\mathbb{P}$-names for elements of $\dot{\mathbb{Q}}$ such that $p \Vdash$"$\dot{X}_i = \{\dot{q}[\dot{G}] \mid \dot{q} \in Y_p^i\}$".
3. $E_i = \{(p, \dot{q}) \mid p \in \sigma(D_0^{\mathbb{P},\dot{X}_0}, D_1^{\mathbb{P},\dot{X}_1}, \ldots, D_i^{\mathbb{P},\dot{X}_i}) \wedge \dot{q} \in Y_p^i\}$.

The game continues in the natural way. Player I plays D_{n+1}, define $\dot{X}_{n+1} = \dot{\pi}(D_0^{\dot{G}}, \ldots, D_{n+1}^{\dot{G}})$, for every $p \in \sigma(D_0^{\mathbb{P},\dot{X}_0}, D_1^{\mathbb{P},\dot{X}_1}, \ldots, D_{n+1}^{\mathbb{P},\dot{X}_{n+1}})$, let Y_p^{n+1} be such that $p \Vdash$ "$\dot{X}_{n+1} = \{\dot{q}[\dot{G}] \mid \dot{q} \in Y_p^{n+1}\}$". Finally, Player II plays $E_{n+1} = \{(p, \dot{q}) \mid p \in \sigma(D_0^{\mathbb{P},\dot{X}_0}, D_1^{\mathbb{P},\dot{X}_1}, \ldots, D_{n+1}^{\mathbb{P},\dot{X}_{n+1}}) \wedge \dot{q} \in Y_p^{n+1}\}$.

We claim that this is a winning strategy for player II. In order to achieve this, we must prove that $(x, \dot{y})$ has an extension in which every E_n is predense. For every $n \in \omega$, let $F_n = \{p \in \mathbb{P} \mid \exists \dot{q}\,(p, \dot{q}) \in E_n\}$. Note that the sequence $\langle F_n \rangle_{n\in\omega}$ corresponds to a run of the game $\mathcal{BG}(\mathbb{P}, x)$ where player I played $D_n^{\mathbb{P},\dot{X}_n}$ at step n (i.e. $F_n = \sigma(D_0^{\mathbb{P},\dot{X}_0}, D_1^{\mathbb{P},\dot{X}_1}, \ldots, D_n^{\mathbb{P},\dot{X}_n})$ for every $n \in \omega$). Since σ is a winning strategy for player II, there is $a \leqslant x$ such that F_n is predense below a for every $n \in \omega$.

Let $G \subseteq \mathbb{P}$ be a generic filter with $a \in G$. For the moment, we will work in the extension $V[G]$. Note that the sequence $\langle \dot{X}_n[G] \rangle_{n\in\omega}$ corresponds to a run of the game $\mathcal{BG}(\dot{\mathbb{Q}}[G], \dot{y}[G])$ where player I played D_n^G at step n. Since $\dot{\pi}[G]$ is a winning strategy for player II, there is $\dot{b}[G] \leqslant \dot{y}[G]$ such that $\dot{X}_n[G]$ is predense below $\dot{b}[G]$ for every $n \in \omega$. We may assume (by extending a if necessary) that $a \Vdash$ "$\dot{b} \leqslant \dot{y}$"and that a forces that each $\dot{X}_n$ if predense below $\dot{b}$. In this way, we have that $(a, \dot{b}) \leqslant (x, \dot{y})$. We will now prove that each E_n is predense below $(a, \dot{b})$.

Let $(p_1, \dot{q}_1) \leqslant (a, \dot{b})$ and $n \in \omega$. Since F_n is predense below a, there is $p_2 \in F_n$ that is compatible with p_1, find $r \leqslant p_1, p_2$. Since $r \leqslant p_2$, we know that $r \Vdash$ "$\dot{X}_n = \{\dot{q} \mid \dot{q} \in Y_{p_2}^n\}$". Now, $\dot{X}_n$ is forced to be predense below $\dot{b}$ (recall that r extends a), so we can find $r_1 \leqslant r$ and $\dot{q}_2 \in Y_{p_2}^n$ such that $r_1 \Vdash$ "$\dot{q}_2 \parallel \dot{q}_1$". Let $\dot{q}$ be a $\mathbb{P}$-name such that $r_1 \Vdash$ "$\dot{q}_3 \leqslant \dot{q}_1, \dot{q}_2$". In this way, we have that $(r_1, \dot{q}_3) \leqslant (p_1, \dot{q}_1)$ and extends an element of E_n (namely, $(p_2, \dot{q}_2)$). □

We will recall a well-known forcing lemma that will be often used implicitly (for a proof, see Lemma 1.19 in the first chapter of [39]). This lemma is often referred as the "definition by cases Lemma":

Lemma 6.4 *Let $\mathbb{P}$ be a partial order, $A = \{p_\alpha \mid \alpha \in \kappa\} \subseteq \mathbb{P}$ a maximal antichain and $\{\dot{x}_\alpha \mid \alpha \in \kappa\}$ be a set of $\mathbb{P}$-names. There is a $\mathbb{P}$-name $\dot{y}$ such that $p_\alpha \Vdash$ "$\dot{y} = \dot{x}_\alpha$" for every $\alpha \in \kappa$.*

We will now prove the preservation at limit steps. Below, if $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leqslant \delta \rangle$ is a countable support iteration and $p \in \mathbb{P}_\delta$, by $\mathrm{sop}(p)$ we denote the support of p.

Theorem 6.5 *Let δ be a limit ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leqslant \delta \rangle$ a countable support iteration of forcings. If $\mathbb{P}_\alpha \Vdash$"$\dot{\mathbb{Q}}_\alpha$ is strategically bounding" for every $\alpha < \delta$, then $\mathbb{P}_\delta$ is strategically bounding.*

Proof By Lemma 6.1, we may assume that each $\dot{\mathbb{Q}}_\alpha$ is forced to be a Boolean algebra. Let $p \in \mathbb{P}_\delta$. We need to prove that player II has a winning strategy in $\mathcal{BG}(\mathbb{P}_\delta, p)$.

Let D_0 be the first move of player I *in* $\mathcal{BG}(\mathbb{P}_\delta, p)$.

$\mathcal{BG}(\mathbb{P}_\delta, p)$

I	D_0	
II		

Let $\gamma_0 = 0$ and find $p_0 \leqslant p$ such that $p_0 \in D_0$. Player II will play $B_0 = \{p_0\}$ as her first move.

Let D_1 be the next move of player I *in* $\mathcal{PG}(\mathbb{P}_\delta, p)$.

$\mathcal{PG}(\mathbb{P}_\delta, p)$

I	D_0		D_1	
II		B_0		

Choose $\gamma_1 \in \mathrm{sop}(p_0)$ with $\gamma_1 \neq \gamma_0$. We have that $\gamma_0 < \gamma_1$.

We now go to V_{γ_1+1}

Let $E_1 = \{q \in \mathbb{P}_\delta/\mathbb{P}_{\gamma_1+1} \mid \exists a \in G_{\gamma_1+1}\ (a^\frown q \in D_1)\}$. It is easy to see that E_1 is an open dense subset of the quotient $\mathbb{P}_\delta/\mathbb{P}_{\gamma_1+1}$. In this way, we can find a condition $q_{10} \leqslant p_0 \restriction (\gamma_1 + 1, \delta)$ such that $q_{10} \in E_1$.

We now go to V_{γ_1}

In here, let $\dot{q}_{10}$ be a $\mathbb{Q}_{\gamma_1}$-name for q_{10}. Define $D_0^{\gamma_1} = \{b \in \mathbb{Q}_{\gamma_1} \mid \exists q_{10}^b (b \Vdash$ "$\dot{q}_{10} = q_{10}^b$")$\}$, which is an open dense subset of $\mathbb{Q}_{\gamma_1}$. Consider a run of the game $\mathcal{BG}(\mathbb{Q}_{\gamma_1}, p_0(\gamma_1))$ in which player I played $D_0^{\gamma_1}$ as his first move. Let $B_0^{\gamma_1}$ be the response of player II (where she is following her winning strategy for this game, recall that $\mathbb{Q}_{\gamma_1}$ was forced to be a strategically bounding forcing).

$\mathcal{BG}(\mathbb{Q}_{\gamma_1}, p_0(\gamma_1))$

I	$D_0^{\gamma_1}$			
II		$B_0^{\gamma_1}$		

Let $B_0^{\gamma_1} = \{a_1, \ldots, a_n\}$ and for every $i \leqslant n$ let q_{10}^i be such that $a_i \Vdash$ "$\dot{q}_{10} = q_{10}^i$". We now define $\widehat{q}_{10}$ a $\mathbb{Q}_{\gamma_1}$-name for an element of $\mathbb{P}_\delta/\mathbb{P}_{\gamma_1+1}$ such that the following conditions hold:

1. $a_1 \Vdash$ "$\widehat{q}_{10} = q_{10}^1$".
2. $a_1^* \wedge a_2 \Vdash$ "$\widehat{q}_{10} = q_{10}^2$". [9]
3. $a_1^* \wedge a_2^* \wedge a_3 \Vdash$ "$\widehat{q}_{10} = q_{10}^3$".

$\vdots$

n. $a_1^* \wedge \ldots \wedge a_{n-1}^* \wedge a_n \Vdash$ "$\widehat{q}_{10} = q_{10}^n$".

$n+1$. $a_1^* \wedge \ldots \wedge a_{n-1}^* \wedge a_n^* \Vdash$ "$\widehat{q}_{10} = p_0 \restriction (\gamma_1 + 1, \delta)$".

This is possible by the "definition by cases lemma". We now define $A_0^{\gamma_1} = \{a_i{}^\frown \widehat{q}_{10} \mid i \leqslant n\}$. Note that $A_0^{\gamma_1} \subseteq \mathbb{P}_\delta/\mathbb{P}_{\gamma_1}$ and every element of it extends $p_0 \restriction (\gamma_1, \delta)$.

[9] If $\mathbb{B}$ is a Boolean algebra and $b \in \mathbb{B}$, we denote by b^* as the complement of b.

We now go to V_{γ_0+1}

In V_{γ_0+1} we may take $q_{11} \leqslant p_0 \upharpoonright (\gamma_0 + 1, \gamma_1)$ such that there is $C_0^{\gamma_1}$ such that $q_{11} \Vdash$ "$\dot{A}_0^{\gamma_1} = C_0^{\gamma_1}$".

We now go to $V_{\gamma_0} = V$

In here, let $\dot{q}_{11}$ be a $\mathbb{Q}_{\gamma_0}$-name for q_{11}. Define $D_0^{\gamma_0} = \{b \in \mathbb{Q}_{\gamma 0} \mid \exists q_{11}^b (b \Vdash$ "$\dot{q}_{11} = q_{11}^b$")$\}$, which is an open dense subset of $\mathbb{Q}_{\gamma 0}$. Consider a run of the game $\mathcal{BG}(\mathbb{Q}_{\gamma_0}, p_0(\gamma_0))$ in which player I played $D_0^{\gamma_1}$ as his first move. Let $B_0^{\gamma_1}$ be the response of player II (where she is following her winning strategy).

$\mathcal{BG}(\mathbb{Q}_{\gamma_0}, p_0(\gamma_0))$

I	$D_0^{\gamma_0}$			
II		$B_0^{\gamma_0}$		

Let $B_0^{\gamma_1} = \{b_1, \ldots, b_m\}$ and for every $i \leqslant m$ let q_{11}^i be such that $b_i \Vdash$ "$\dot{q}_{11} = q_{11}^i$". We now define $\widehat{q}_{11}$ a $\mathbb{Q}_{\gamma_0}$-name for an element of $\mathbb{P}_{\gamma_1}/\mathbb{P}_{\gamma_0+1}$ such that the following conditions hold:

1. $b_1 \Vdash$ "$\widehat{q}_{11} = q_{11}^1$".
2. $b_1^* \wedge b_2 \Vdash$ "$\widehat{q}_{11} = q_{11}^2$".

$\vdots$

n. $b_1^* \wedge \ldots \wedge b_{m-1}^* \wedge b_m \Vdash$ "$\widehat{q}_{11} = q_{11}^m$".
$n+1$. $b_1^* \wedge \ldots \wedge b_{n-1}^* \wedge b_m^* \Vdash$ "$\widehat{q}_{11} = p_0 \upharpoonright (\gamma_0 + 1, \gamma_1)$".

We now define $A_0^{\gamma_1} = \{b_i {}^\frown \widehat{q}_{11} \mid i \leqslant m\}$. Note that $A_0^{\gamma_1} \subseteq \mathbb{P}_\delta/\mathbb{P}_{\gamma_1}$ and every element of it extends $p_0 \upharpoonright (\gamma_1, \delta)$. Now, let B_1 be the set of all elements of the form

$$b {}^\frown \widehat{q}_{11} {}^\frown a$$

such that

1. $b \in B_0^{\gamma_0}$.
2. $b {}^\frown \widehat{q}_{11} \Vdash$ "$a \in \dot{A}_0^{\gamma_1}$".

Finally, player II plays B_1 in $\mathcal{BG}(\mathbb{P}_\delta, p)$.

$\mathcal{BG}(\mathbb{P}_\delta, p)$

I	D_0		D_1		
II		B_0		B_1	

Let $p_2 = p_1(\gamma_0) {}^\frown \widehat{q}_{11} {}^\frown p_1(\gamma_1) {}^\frown \widehat{q}_{10}$. Note that $p_1(\gamma_0)$ knows a countable superset of the support of $\widehat{q}_{11}$, while $p_1(\gamma_0) {}^\frown \widehat{q}_{11} {}^\frown p_1(\gamma_1)$ knows a countable superset of the support of $\widehat{q}_{10}$. This is the reason why although formally p_2 is a condition in $\mathbb{Q}_{\gamma_0} * \mathbb{P}_{(\gamma_0,\gamma_1)} * \mathbb{Q}_{\gamma_1} * \mathbb{P}_{(\gamma_1,\delta)}$, we may identify it with a condition in $\mathbb{P}_\delta$. Let D_2 be the next move of player I in $\mathcal{BG}(\mathbb{P}_\delta, p)$.

$\mathcal{BG}(\mathbb{P}_\delta, p)$

I	D_0		D_1		D_2		
II		B_0		B_1			

Choose $\gamma_2 \in \text{sop}(p_1)$ with $\gamma_2 \notin \{\gamma_0, \gamma_1\}$. For convenience, assume that $\gamma_0 < \gamma_2 < \gamma_1$.

We now go to V_{γ_1+1}

Let $E_2 = \{q \in \mathbb{P}_\delta/\mathbb{P}_{\gamma_1+1} \mid \exists\, a \in G_{\gamma_{1+1}}\, (a^\frown q \in D_2)\}$. It is easy to see that E_2 is an open dense subset of the quotient $\mathbb{P}_\delta/\mathbb{P}_{\gamma_1+1}$. In this way, we can find a condition $q_{20} \leqslant p_2{\restriction}(\gamma_1 + 1, \delta)$ such that $q_{20} \in E_2$.

We now go to V_{γ_1}

In here, let $\dot{q}_{20}$ be a $\mathbb{Q}_{\gamma_1}$-name for q_{20}. Define $D_1^{\gamma_1} = \{b \in \mathbb{Q}_{\gamma_1} \mid \exists\, q_{20}^b\, (b \Vdash \text{“}\dot{q}_{20} = q_{20}^b\text{”})\}$, which is an open dense subset of $\mathbb{Q}_{\gamma_1}$. Let player I play $D_1^{\gamma_1}$ as his next move in $\mathcal{BG}(\mathbb{Q}_{\gamma_1}, p_0(\gamma_1))$. Let $B_1^{\gamma_1}$ be the response of player II (where she is following her winning strategy).

$\mathcal{BG}(\mathbb{Q}_{\gamma_1}, p_0(\gamma_1))$

I	$D_0^{\gamma_1}$		$D_1^{\gamma_1}$	
II		$B_0^{\gamma_1}$		$B_1^{\gamma_1}$

Let $B_1^{\gamma_1} = \{a_1, \ldots, a_n\}$ and for every $i \leqslant n$ let q_{20}^i be such that $a_i \Vdash \text{“}\dot{q}_{20} = q_{20}^i\text{”}$. We now define $\widehat{q}_{20}$ a $\mathbb{Q}_{\gamma_1}$-name for an element of $\mathbb{P}_\delta/\mathbb{P}_{\gamma_1+1}$ such that the following conditions hold:

1. $a_1 \Vdash \text{“}\widehat{q}_{20} = q_{20}^1\text{”}$.
2. $a_1^* \wedge a_2 \Vdash \text{“}\widehat{q}_{20} = q_{20}^2\text{”}$.
3. $a_1^* \wedge a_2^* \wedge a_3 \Vdash \text{“}\widehat{q}_{20} = q_{20}^3\text{”}$.

$\vdots$

n. $a_1^* \wedge \ldots \wedge a_{n-1}^* \wedge a_n \Vdash \text{“}\widehat{q}_{20} = q_{20}^n\text{”}$.
$n+1$. $a_1^* \wedge \ldots \wedge a_{n-1}^* \wedge a_n^* \Vdash \text{“}\widehat{q}_{20} = p_2{\restriction}(\gamma_1 + 1, \delta)\text{”}$.

We now define $A_1^{\gamma_1} = \{a_i{}^\frown\widehat{q}_{10} \mid i \leqslant n\}$. Note that $A_1^{\gamma_1} \subseteq \mathbb{P}_\delta/\mathbb{P}_{\gamma_1}$ and every element of it extends $p_2{\restriction}(\gamma_1, \delta)$.

We now go to V_{γ_2+1}

In V_{γ_2+1} we may take $q_{21} \leqslant p_2{\restriction}(\gamma_2 + 1, \gamma_1)$ such that there is $C_1^{\gamma_1}$ such that $q_{11} \Vdash \text{“}\dot{A}_1^{\gamma_1} = C_1^{\gamma_1}\text{”}$.

We now go to V_{γ_2}

In here, let $\dot{q}_{21}$ be a $\mathbb{Q}_{\gamma_2}$-name for q_{21}. Define $D_0^{\gamma_2} = \{b \in \mathbb{Q}_{\gamma 0} \mid \exists\, q_{21}^b\, (b \Vdash \text{“}\dot{q}_{21} = q_{21}^b\text{”})\}$, which is an open dense subset of $\mathbb{Q}_{\gamma 0}$. Consider a run of the game $\mathcal{BG}(\mathbb{Q}_{\gamma_2}, p_2(\gamma_2))$ in which player I played $D_0^{\gamma_2}$ as his first move. Let $B_0^{\gamma_2}$ be the response of player II (where she is following her winning strategy).

$\mathcal{BG}(\mathbb{Q}_{\gamma_2}, p_2(\gamma_2))$

I	$D_0^{\gamma_2}$			
II		$B_0^{\gamma_2}$		

Let $B_0^{\gamma_2} = \{b_1, \ldots, b_m\}$ and for every $i \leqslant m$ let q_{21}^i be such that $b_i \Vdash$ "$\dot{q}_{21} = q_{21}^i$". We now define $\widehat{q}_{21}$ a $\mathbb{Q}_{\gamma_2}$-name for an element of $\mathbb{P}_{\gamma_1}/\mathbb{P}_{\gamma_2+1}$ such that the following conditions hold:

1. $b_1 \Vdash$ "$\widehat{q}_{21} = q_{21}^1$".
2. $b_1^* \wedge b_2 \Vdash$ "$\widehat{q}_{21} = q_{21}^2$".

$\vdots$

m. $b_1^* \wedge \ldots \wedge b_{m-1}^* \wedge b_m \Vdash$ "$\widehat{q}_{21} = q_{21}^m$".
$m+1$. $b_1^* \wedge \ldots \wedge b_{m-1}^* \wedge b_m^* \Vdash$ "$\widehat{q}_{21} = p_2 \restriction (\gamma_2 + 1, \gamma_1)$".

We now define $A_0^{\gamma_2} = \{b_i {}^\frown \widehat{q}_{21} \mid i \leqslant m\}$. Note that $A_0^{\gamma_2} \subseteq \mathbb{P}_{\gamma_1}/\mathbb{P}_{\gamma_2}$ and every element of it extends $p_2 \restriction (\gamma_2, \gamma_1)$.

We now go to V_{γ_0+1}
In V_{γ_0+1} we may take $q_{22} \leqslant p_2 \restriction (\gamma_0 + 1, \gamma_2)$ such that there is $C_0^{\gamma_2}$ such that $q_{22} \Vdash$ "$\dot{A}_0^{\gamma_2} = C_0^{\gamma_2}$".

We now go to $V_{\gamma_0} = V$
In here, let $\dot{q}_{22}$ be a $\mathbb{Q}_{\gamma_0}$-name for q_{22}. Define $D_1^{\gamma_0} = \{b \in \mathbb{Q}_{\gamma_0} \mid \exists\, q_{22}^b (b \Vdash$ "$\dot{q}_{22} = q_{22}^b$")$\}$, which is an open dense subset of $\mathbb{Q}_{\gamma_0}$. Let player I play $D_1^{\gamma_0}$ as his next move in $\mathcal{BG}(\mathbb{Q}_{\gamma_0}, p_0(\gamma_0))$. Let $B_1^{\gamma_0}$ be the response of player II (where she is following her winning strategy).

$\mathcal{BG}(\mathbb{Q}_{\gamma_0}, p_0(\gamma_0))$

I	$D_0^{\gamma_0}$		$D_1^{\gamma_0}$			
II		$B_0^{\gamma_0}$		$B_1^{\gamma_0}$		

Let $B_1^{\gamma_0} = \{c_1, \ldots, c_l\}$ and for every $i \leqslant l$ let q_{22}^i be such that $c_i \Vdash$ "$\dot{q}_{22} = q_{22}^i$". We now define $\widehat{q}_{22}$ a $\mathbb{Q}_{\gamma_0}$-name for an element of $\mathbb{P}_{\gamma_2}/\mathbb{P}_{\gamma_1+1}$ such that the following conditions hold:

1. $c_1 \Vdash$ "$\widehat{q}_{11} = q_{22}^1$".
2. $c_1^* \wedge c_2 \Vdash$ "$\widehat{q}_{22} = q_{22}^2$".

$\vdots$

l. $c_1^* \wedge \ldots \wedge c_{l-1}^* \wedge c_l \Vdash$ "$\widehat{q}_{22} = q_{22}^l$".
$l+1$. $c_1^* \wedge \ldots \wedge c_{l-1}^* \wedge c_l^* \Vdash$ "$\widehat{q}_{22} = p_2 \restriction (\gamma_0 + 1, \gamma_2)$".

We now define $A_0^{\gamma_2} = \{c_i {}^\frown \widehat{q}_{22} \mid i \leqslant l\}$. Note that $A_0^{\gamma_1} \subseteq \mathbb{P}_\delta/\mathbb{P}_{\gamma_1}$ and every element of it extends $p_2 \restriction (\gamma_0, \gamma_2)$. Now, let B_2 be the set of all elements of the form

$$c {}^\frown \widehat{q}_{22} {}^\frown b {}^\frown a$$

such that

1. $c \in B_1^{\gamma_0}$.

2. $c^\frown \widehat{q}_{22} \Vdash$ "$b \in \dot{A}_0^{\gamma_2}$".
3. $c^\frown \widehat{q}_{22}{}^\frown b \Vdash$ "$\dot{A}_1^{\gamma_1}$".

Finally, player II plays B_2 in $\mathcal{BG}(\mathbb{P}_\delta, p)$.

$\mathcal{BG}(\mathbb{P}_\delta, p)$

I	D_0		D_1		D_2		
II		B_0		B_1		B_2	

Let $p_3 = p_2(\gamma_0)^\frown \widehat{q}_{22}{}^\frown p_2(\gamma_2)^\frown \widehat{q}_{21}{}^\frown p_2(\gamma_1)^\frown \widehat{q}_{20}$.

The game continues in this way. Furthermore, by carefully choosing each γ_n, we make sure that $\bigcup_{n\in\omega} \mathrm{sop}(p_n) = \{\gamma_n \mid n \in \omega\}$. We now define a condition $r_1 \in \mathbb{P}_\delta$ with the following properties:

1. $\mathrm{sop}(r_1) = \{\gamma_n \mid n \in \omega\}$.
2. $r_1(\gamma_n) = p_n(\gamma_n)$.

Note that r_1 extends each p_n. For every $n \in \omega$, the sequence $\langle D_i^{\gamma_n}, B_i^{\gamma_n}\rangle_{i\in\omega}$ is (forced to be) a run of the game in $\mathcal{BG}(\mathbb{Q}_{\gamma_n}, r_1(\gamma_n))$ in which player II followed her winning strategy. It follows that there must be a (name for a condition) that forces that each $B_i^{\gamma_n}$ is predense below it. Define $r \in \mathbb{P}_\delta$ with $\mathrm{sop}(r) = \{\gamma_n \mid n \in \omega\}$ and for each $n \in \omega$, we have that $r(\gamma_n)$ is a name for a condition given by the game $\mathcal{BG}(\mathbb{Q}_{\gamma_n}, r_1(\gamma_n))$. It follows that r extends r_1. It is easy to see that player II was the winner in the $\mathcal{BG}(\mathbb{P}_\delta, p)$. □

By Propositions 6.3 and 6.5, we conclude the following:

Corollary 6.6 *Let δ be an ordinal and $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leqslant \delta\rangle$ a countable support iteration of forcings. If $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is strategically bounding" for every $\alpha < \delta$, then $\mathbb{P}_\delta$ is strategically bounding.*

7 Open questions

In this last section, we list some problems that the authors do not know how to answer. First, the problems of Roitman and of Brendle and Raghavan are still open:

Problem 7.1 (Roitman) Does $\mathfrak{d} = \omega_1$ imply $\mathfrak{a} = \omega_1$?

Problem 7.2 (Brendle, Raghavan [8]) Does $\mathfrak{b} = \mathfrak{s} = \omega_1$ imply $\mathfrak{a} = \omega_1$?

Regarding the theory of strategically bounding forcings, we do not know the following:

Problem 7.3 Let $\mathbb{P}$ and $\mathbb{Q}$ be two forcing equivalent partial orders. If $\mathbb{P}$ is strategically bounding, does it follow that $\mathbb{Q}$ is strategically bounding?

In [35] the diamond principle $\Diamond(\mathfrak{d})$ was introduced, which is strictly stronger than $\Diamond_{\mathfrak{d}}$. We can ask the following:

Problem 7.4 Let $V \models \Diamond$ and $\mathbb{P}$ be a strategically bounding forcing. Does $\Diamond(\mathfrak{d})$ hold after forcing with $\mathbb{P}$?

Recall that if $\mathbb{P}$ is strategically bounding, then its Boolean completion is also strategically bounding, which is a partial answer to the problem above.

Problem 7.5 Is there a strategically bounding forcing that is not an Axiom A forcing for $\mathfrak{d}$?

It is known that there are two forcing equivalent partial orders where one is Axiom A and the other is not, so it is likely that this problem has a positive answer.

Problem 7.6 If $\mathbb{P}$ is strategically bounding, is there an Axiom A forcing for $\mathfrak{d}$ that is forcing equivalent to $\mathbb{P}$?

Recall that by a theorem of Ishiu (see [25]) every $< \omega_1$-proper forcing has an Axiom A representation, so it is possible that this problem has a positive answer.

Regarding tree MAD-families, we do not know the following:

Problem 7.7 Is $\mathfrak{a} \leqslant \mathfrak{a}_T$?

A negative answer seems very hard to obtain at the present knowledge. A model of $\mathfrak{a}_T < \mathfrak{a}$ obtained by iterating proper forcings (over a model of CH) would also be a model where the problem of Roitman is solved. On the other hand, other methods like iterating along a template does not seem to help either.

Problem 7.8 Is it consistent that there are no tree Shelah–Steprāns MAD families?

Recall that Raghavan constructed a model where there are no Shelah–Steprāns MAD families.

Problem 7.9 Is there a combinatorial characterization of the tree-MAD families $\mathcal{A}$ such that the Miller forcing $\mathbb{P}(\mathcal{A})$ is not strategically bounding? Does ZFC imply the existence of such families?

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